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AN ANALYSIS OF THE J-PHENOMENON IN X-RAYS

Part I.

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ABSTRACT. In this paper an attempt has been made to study some aspects of the *J*-phenomenon in X-rays from an analytical point of view. Some of the principal features of the phenomenon as well as the conditions governing the occurrence of the 'J-discontinuity' in two kinds of experiments have been discussed. It has been shown from theoretical considerations that the effect together with the associated features, is exactly what is expected under suitable circumstances. Barkla's concept of 'levels' of X-ray activity has not been considered in the present analysis.

INTRODUCTION

It is well known that Barkla and his collaborators had observed under certain conditions discontinuities in X-ray absorption which at first were considered as an evidence of a *J*-series of characteristic *X*-radiation. It is also well known that later investigations led Barkla to discard this view but many incontrovertible facts were there which baffled explanation and Barkla classed them under the name, "*J*-phenomenon in *X*-rays".

The experimental results of Barkla *et al.* relate mainly to the absorption of heterogenous X-rays by matter. The two methods which were usually followed may be described as follows :

I. The mass-absorption coefficient of a heterogeneous beam of X-rays, from which the softer constituents had been eliminated by filtration, was determined as usual in two testing substances. The penetrating power of the incident beam was progressively increased and the mass-absorption coefficients in two substances were measured for each penetrating power. They were then plotted, one against the other in a graph.

II. A heterogeneous beam of X-rays was progressively hardened by passing it through an increasing thickness of a sheet of substance. The mass-absorption coefficient of the transmitted beam was then determined in either a similar substance or in a different substance and was plotted in a graph against the percentage of the incident radiation absorbed by the filter.

In the above two methods, the mass-absorption coefficient of the heterogeneous beam in any testing substance was measured in the usual way, i.e., by placing in its path, sheets of the testing substance and adjusting their thickness

till there was a 50% reduction in the intensity of the beam. If t represented the intercepting thickness, then the average mass-absorption coefficient would be given by $\left(\frac{\bar{\mu}}{\rho}\right) = \frac{1}{t}\log_e 2$. The range of values for $\left(\frac{\bar{\mu}}{\rho}\right)_{AL}$ employed in the experiments under review was generally between 0.5 and 1.8 approximately, the corresponding wavelengths being nearly 0.29Å and 0.48Å respectively.

In both these methods, the graphs obtained can be classified as two distinct *alternative* cases, Cases *A* and *B*. In Case B, the graph was a continuous straight line and in Case A there were two and sometimes more, straight lines intervened by one or more discontinuity or discontinuities. The discontinuity or discontinuities observed in Case A constituted a notable feature of Barkla's *J*-phenomenon in X-rays.

The *J*-discontinuities had been observed times without number under various experimental conditions. It had been observed with an intense, narrow pencil of primary rays, or a weak, wide pencil of scattered rays; with heterogeneous radiations or approximately homogeneous radiations; with highly polarized or unpolarized radiations; with the same tube, but rays proceeding along different directions or with different tubes excited in similar or different ways. It had been observed by different investigators with different testing substances and by various testing methods. Again, under precisely the same conditions, it had not been observed by others and even by the same observers.

The principal features of the discontinuity-phenomenon may be outlined as follows :

(i) The discontinuity-phenomenon, when it occurred, occurred at a critical penetrating power of the radiation (as measured by the mass-absorption coefficient) which was characteristic of the testing substance and independent of the previous history of the radiation. The critical penetrating power changed slightly with the material of the testing element.

(ii) A number of discontinuities called J_1, J_2, J_3, ... etc., each at its own critical frequency, was observed.

(iii) The discontinuity was very abrupt.

(iv) The discontinuity or discontinuities depended on some factors which could not be identified.

Amongst those who reported failures to reproduce the J-discontinuity were Dunbar (1925, 1928), Worsnop (1927), Alexander (1930), Backhurst (1932) and a few others. Their results represented only one of the two alternative cases (Case B) and could add nothing to our knowledge. Such a result was neither more nor less real and hence could not adequately prove or disprove the phenomenon itself.

In order to explain the phenomenon, Barkla had introduced a new concept of two (or more) discrete 'levels' of X-ray activity (absorption) and postulated sudden transformation, under proper circumstances, from one 'level' to another, depending upon the complex heterogeneous beam taken as a whole. This concept was not however acceptable. Barkla himself had admitted a number of times that much more experimental work would be necessary for any final explanation which would be convincing.

In the present investigations, an attempt has been made, on the basis of the heterogeneity of the X-ray beam, to give a consistent and comprehensive interpretation of all the experimental results which had been obtained by Barkla and his collaborators. It is significant that it has not been necessary to invoke Barkla's new concept in the interpretation given by the author. The present paper deals with the results of experimental investigations, embodied in the paper on "The *J*-phenomenon in X-rays—Part I" by Barkla (1925).

THEORETICAL CONSIDERATION

A mathematical treatment of the problems on heterogeneous X-rays, is extremely difficult in view of the fact that the beam concerned covers a wide range of wavelength lying between a known lower limit and an uncertain upper limit and also because the energy-distribution over this range is not known with any degree of precision. Besides, the distribution function itself may also vary from tube to tube, depending on many factors.

Truly speaking, there is no one wavelength for a heterogeneous beam. Yet for the sake of convenient reference, it is customary to assign to it, what is called an 'equivalent wavelength', based on its mass-absorption coefficient in some substance, as though the beam were monochromatic. This is not entirely satisfactory. As the absorption coefficient itself is determined somewhat arbitrarily (from a 50% reduction in the intensity) and as it varies with the nature of the testing element, the wavelength deduced from it should naturally be arbitrary to some extent and dependent on the testing material. However, the most important point for consideration is the fact that by this process of measurement, the very structure of the radiation is changed and the change is different with different testing materials. Thus the spectrum of the radiation emerging from the testing substance is different in minute details from that incident on it. As the emergent beam is richer in harder rays on account of the greater absorption of the softer constituents by the material of the tester, the average or the 'equivalent' wavelength on the emergent side is definitely shorter than on the incident side. Under the circumstances, the question naturally arises : To which portion of the beam, the incident or the emergent, should we ascribe the mass-absorption coefficient determined experimentally? Although the usual convention is to attribute it to the incident portion, the emergent one has an equal claim upon it. Such a

problem, of course, does not arise in the case of a perfectly homogeneous beam, where the emergent beam is identical with the incident beam, except that it is of less intensity. A little reflection will suggest that, for a heterogeneous beam, the mass-absorption coefficient should belong neither to the incident side, nor to the emergent. More appropriately it should correspond to an intermediate wavelength which the beam has somewhere inside the testing absorber.

The Effective Wavelength of the heterogeneous radiation

This intermediate wavelength, which we shall call 'effective' wavelength λ_{eff} can be interpolated from the following relation holding good between the mass-absorption coefficient and the wavelength of a monochromatic radiation, lying within the range of wavelengths with which we are concerned :

$$\frac{\mu}{\rho} = A + B\lambda^3 \qquad \ldots \quad (1)$$

This is a linear relationship between μ/ρ and λ^3, where A and B are constants depending on the nature of the testing material. The constant A may also depend upon the geometry of the measuring arrangement. The approximate values of A and B, computed for aluminium and copper from the standard data obtained with a narrow pencil of X-rays, may be written down as follows :

Aluminium	$A = 0.125$	$B = 14.1$
Copper	$A = 0$	$B = 153$

The 'Discrepancy' between the Effective Wavelength and the Average Wavelength.

We have now to probe into the relation corresponding to (1) between the measured (μ/ρ) and the average wavelength λ' of the incident spectrum of a heterogeneous beam of X-rays. This 'average' wavelength is to be regarded as independent* of the testing material. Obviously the 'effective" wavelength λ_{eff} falls short of the "average" wavelength λ' by a quantity 'e' so that

$$\lambda_{eff} = \lambda' - e \qquad \ldots \quad (2)$$

Let e be called the 'discrepancy', which is necessarily a function of λ' and the nature of the absorbing material. For a monochromatic radiation, e is equal to zero, and for hard, filtered and heterogeneous radiations (such as were employed in the experiments under review) it is calculated to be small. Further, so long as the average wavelength of the latter remains unchanged, slight variations in their microscopic structure will be supposed to produce only a trivial change, if at all, in e—a change of second-order smallness which will be neglected.

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*Perhaps the 'average' wavelength λ' here may be best defined as follows : $\lambda' = \frac{c}{\nu'}$ where the average frequency $\nu' = (\Sigma\, n.h.\, \nu/(h.\Sigma n) =$ Total energy$/(h \times$ total number of photons).

Now, λ_{eff} satisfies the equation :

$$\frac{\bar{\mu}}{\rho} = A + B\lambda^3_{eff}, \text{ which in view of (2)}$$

becomes :

$$\frac{\bar{\mu}}{\rho} = A + B(\lambda' - e)^3$$

$$= A + B\lambda'^3(1 - 3e/\lambda'), \text{ since } e << \lambda'$$

Deleting the dash upon λ for the sake of convenience, we shall henceforth write :

$$\frac{\bar{\mu}}{\rho} = A + B\lambda^3(1 - 3e/\lambda) \qquad \ldots \quad (3)$$

where λ means now and hereafter the 'average' wavelength of the incident radiation. Putting $\bar{\mu}/\rho = y$ and $\lambda^3 = x$, equation (3) takes the form

$$y = A + Bx - 3Bx^{2/3}e \qquad \ldots \quad (3a)$$

Thus what was a straight line graph (*vide* equation 1) in the case of a monochromatic radiation becomes a curve* in the case of a heterogeneous radiation, the equation to the curve being given by 3(a). Fig. 1 represents schematically the two graphs one above another.

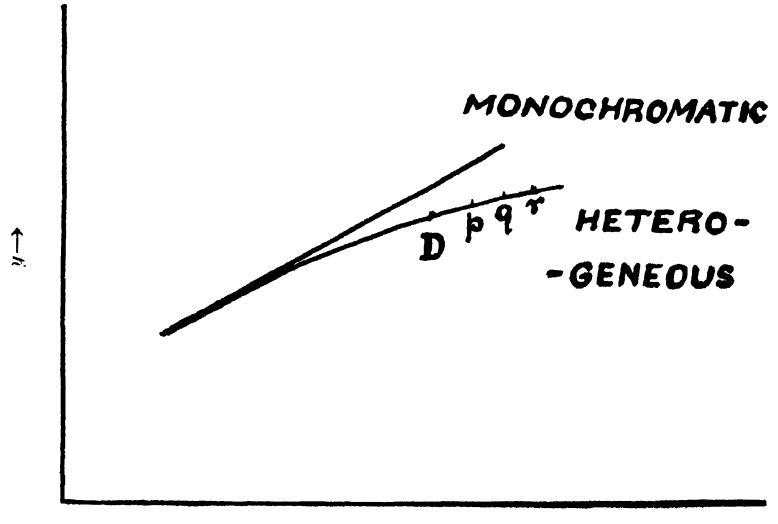

Fig. 1

Curvature of the graph for the heterogeneous beam

Differentiating equation (3a) successively with respect to x, we get :

$$y_1 = B(1 - 3x^{2/3}e_1 - 2x^{-1/3}e) \qquad (4)$$

*It should be stressed here, that the deviation of this curve from the linear course is nowhere great within the range of experimental wavelengths, as the term $3e/\lambda$ in equation (3) is small compared to unity.

$$y_2 = -\frac{B}{3}(9x^{2/3}e_2 + 12x^{-1/3}e_1 - 2x^{-4/3}e) \quad \ldots (5)$$

and

$$y_3 = -Bx^{-7/3}\left(3x^3e_3 + 6x^2e_2 - 2xe_1 + \frac{8}{9}e\right) \quad \ldots (6)$$

These differential coefficients cannot ordinarily be evaluated since e, as a function of x, is not known. In the present analysis, we shall suppose that for filtered, hard radiation,

$$e = ax + bx^2 + cx^3 \quad \text{(approx.)} \quad \ldots (7)$$

where a, b, c are positive constants, independent of x but dependent on the testing absorber.

Differentiating now equation (7), we have

$$\left.\begin{aligned} \frac{de}{dx} &= e_1 = a + 2bx + 3cx^2 \\ \frac{d^2e}{dx^2} &= e_2 = 2b + 6cx \\ \frac{d^3e}{dx^3} &= e_3 = 6c \end{aligned}\right\} \quad \ldots (8)$$

A justification for the function expressed by equation (7) arises from the fact that it satisfies the conditions that both e and de/dx should decrease with the decrease of λ, (i.e., with the decrease of x) in conformity with the actual behaviour of the rays concerned.

Substituting in equation (5) the values of e and the differential coefficients from (7) and (8) respectively, we obtain after simplification,

$$y_2 = -\frac{2B}{3}(5ax^{-1/3} + 20bx^{2/3} + 44cx^{5/3}) \quad (9)$$

It is noted from this equation (9) that y_2 is negative, so that the slope of the curve represented by (3a) decreases as x increases. Further, the rate of variation of the slope is comparatively great for small as well as for large values of x (i.e. of λ). Hence this rate passes through a minimum at a certain medium wave length. If this minimum occurs within the experimental range of wave-length, a J-discontinuity will appear. The particular value of x at which such ami nimum occurs can be calculated by putting $y_3 = 0$ in equation (6), so that

$$0 = y_3 = -Bx^{-7/3}\left(3x^3e_3 + 6x^2e_2 - 2xe_1 + \frac{8}{9}e\right)$$

or

$$3x^3e_3 + 6x^2e_2 - 2xe_1 + \frac{8}{9}e = 0 \quad \ldots (10)$$

The solution of the equation (10) gives the required value of x. The point on the curve corresponding to this minimum is denoted by D.

DEDUCTIONS

1. *Monochromatic radiation.*

From equation (1) we have

$$\left(\frac{\mu}{\rho}\right)_X = A_X + B_X\lambda^3$$

and

$$\left(\frac{\mu}{\rho}\right)_Y = A_Y + B_Y\lambda^3,$$

where the subscripts X and Y refer to two different testing substances. Eliminating λ^3 between these equations,

$$\left(\frac{\mu}{\rho}\right)_X = \left(A_Y - \frac{B_X}{B_Y}A_Y\right) + \frac{B_X}{B_Y}\left(\frac{\mu}{\rho}\right)_Y \qquad (11)$$

This is a continuous linear relation between $(\mu/\rho)_X$ and $(\mu/\rho)_Y$. If in a particular case, X represents aluminium and Y copper, then $A_Y \simeq 0$,

so that

$$\left(\frac{\mu}{\rho}\right)_{Al} = A_{Al} + \frac{B_{Al}}{B_{Cu}}\left(\frac{\mu}{\rho}\right)_{Cu}.$$

2. *Heterogeneous radiations*

CASE I.

Relation between the mass-absorption coefficients in two absorbing substances when the incident beam is progressively hardened

We have already seen that for a particular assumed functional relation between e and x in the case of a penetrating filtered radiation, there occurs a point D on the $y-x$ curve at which the rate of decrease of the slope is minimum. If such is actually the case under the unknown and correct functional relation also, the rate of variation in the slope close to the point D and on either side of it, will also be near the minimum. A set of points p, q, r may, therefore, be taken on the curve close to and on one side of D, such that they may be regarded as practically lying on a straight line of short length. Similarly, we may take another set of points on the other side of D and they lie on another straight line of short length.

Now consider together the two curves $(y_X - x)$ and $(y_Y - x)$ for the two absorbers X and Y with their own D-points, D_X and D_Y. We shall suppose here that D_X and D_Y are sensibly one above the other. Taking the short straight lines passing through the points p, q, r lying on the same side (L.H.S., say) of

D_X and D_Y in the two curves, it is shown below that, over a small range, the relation between y_X and y_Y is linear, when x is eliminated between them. It is evident from equation (3) that the assumption of a short length of the curve being practically straight means that e/λ is fairly constant with a mean value K over that length, so that on the L.H.S. of the D-points we have :

$$y_X = \left(\frac{\bar{\mu}}{\rho}\right)_X = A_X + B_X\lambda^3\left(1-\frac{3e_X}{\lambda}\right) = A_X + B_X(1-3K_X)x \qquad \ldots \text{(11a)}$$

and $$y_Y = \left(\frac{\bar{\mu}}{\rho}\right)_Y = 0 + B_Y\lambda^3\left(1-\frac{3e_Y}{\lambda}\right) = B_Y(1-3K_Y)x$$

where X is taken to represent aluminimum and Y copper.

Eliminating x between the above two equations, we have,

$$y_X = A_X + \frac{B_X}{B_Y}\, y_Y\{1-3(K_X-K_Y)\}$$

where K_X and K_Y are small compared to unity. Putting $\alpha = (K_X-K_Y) =$ const., we get

$$y_X = A_X + \frac{B_X}{B_Y}(1-3\alpha)y_Y \qquad (12)$$

This equation represents a straight line graph for y_X plotted against y_Y. The slope of the straight line is given by $\frac{B_X}{B_Y}(1-3\alpha)$ and the intercept on the y_X-axis by A_X.

Similarly, on the R. H. S. of the D-points on the two curves, we have another straight line represented by the equation

$$y_x = A_X + \frac{B_X}{B_Y}(1-3\beta)y_Y \qquad (13)$$

where $\beta = (K'_X - K'_Y) =$ const. The slope of this straight line is given by $\frac{B_X}{B_Y}(1-3\beta)$, and the intercept on the y-axis, again by A_X.

If α and β are different, equations (12) and (13) represent two different straight lines with different slopes but equal intercepts.

It is not possible to say which of α and β is greater. If however we assume $\beta > \alpha$, the two straight lines agree with Barkla's in their configurations : The former straight line is on the shorter wavelength side and the latter on the longer wavelength side of the D-points, there being a discontinuity in the region corresponding to these points. This is the well known 'J-discontinuity' which is illustrated in Fig. 2. In case if one D-point D_X is within the experimental range of wavelengths and the other D_Y outside but not far away then also J-step may appear. Here $\alpha = (K_X-K_Y)$ and $\beta = (K'_X-K_Y)$.

A point in support of the theory advanced may be noted here. The experimental value of the intercept A_X for aluminium (obtained from Barkla's Figs. 1, 4 and 9 by extrapolation) agrees remarkably well with the theoretical, which is about 0.125. As for the other testing substances such as paper or gold, reliable data for comparison are not available. Further, the smallness of α and β can be verified if their values are estimated from the slopes of the two straight lines with the help of equations (12) and (13). For example, in the case of Al-Cu pair, in Barkla's Fig. 1, the values of α and β are computed as $-.033$ and $+.023$ respectively.

Position of the J-discontinuity

According to the present view, the position of the J-discontinuity in the graph (Fig. 2), is determined by the D-point. To get the x-coordinate of this point we have to solve equation (10), which is :

$$3x^3e_3+6x^2e_2-2xe_1+\frac{8}{9}e=0$$

Assuming the functional relation given in (7) and substituting in equation (10), the values of e and its differential coefficients from (7) and (8) respectively, we get, after simplification,

$$a+8bx+44cx^2=0 \qquad .. \quad (14)$$

This leads to

$$x=(-2b\pm\sqrt{4b^2+11ac})/22c$$

Since the coefficients a, b, c are all positive, the negative sign before the radical is inadmissible, for that would make x negative, which is absurd. Hence

$$x_D=(-2b+\sqrt{4b^2+11ac})/22c \qquad (15)$$

An approximate solution of equation (14) might at once be obtained, if we could neglect the term $44cx^2$ which is likely to be small as compared to the other two terms. In that case

$$x_D\simeq a/8b, \quad \text{for the absorber } X$$

and

$$x_D'\simeq a'/8b', \quad \text{for the absorber } Y \qquad (16)$$

It is evident that D_X and D_Y will lie one above the other in Fig. 1, to the extent the values of a/b and a'/b' approach each other and the position of the J-discontinuity in Fig. 2, will be determined accordingly.

Critical Mass-absorption Coefficient for the J-discontinuity

It is noted from equation (15) that the position of the D-point should be

characteristic of the absorbing material, since the constants a, b, c are characteristic of that material. This explains why the discontinuity in Fig. 2, should occur

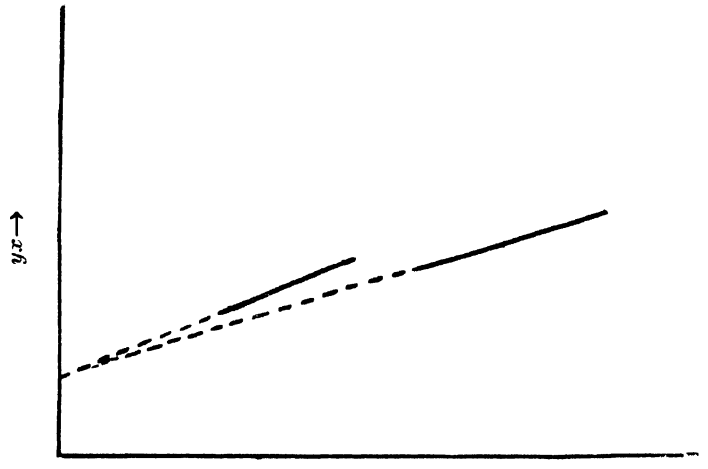

Fig. 2

at a critical value of (μ/ρ) for the absorber concerned. The fact that the J-discontinuity appears at some critical value of (μ/ρ) of the absorber is a characteristic feature of the J-phenomenon. This critical value may be calculated with help of equation (3a), i.e.,

$$\left(\frac{\mu}{\rho}\right)_j = y_j = A + Bx_j - 3Bx_j e_j \qquad \ldots \quad (17)$$

where the subscript 'j' refers to the J-discontinuity and where

$$e_j = ax_j + bx_j^2 + cx_j^3$$

and

$$x_j = x_D = (-2b + \sqrt{4b^2 + 11ac})/22c$$

obtained from equations (7) and (15) respectively. Thus $(\mu/\rho)_j$ turns out to be a function of a, b, and c, which again depend on the absorbing substance. As the nature of variation of a, b, c with the material of the substance is not known due to insufficiency of available data, it is not possible to find how λ_j should change with the testing material. But Barkla's experimental values, $\lambda_{j_3} = 0.335$ Å for aluminium and 0.315 Å for copper, enable us to compute roughly, from equation (16), the ratio a/b for these elements. They are 0.3 for aluminium and 0.25 for copper. It may be mentioned here that on the basis of the function proposed in (7), the J-discontinuity, when it occurs, should appear at a *medium* wavelength. Actually in Barkla's experiments, the discontinuity was found to be within the experimental range, 0.29 Å - 0.48 Å, and the value of the critical wavelength at which the discontinuity appeared varied from 0.30 Å (gold) to 0.39 Å(carbon).

Constancy of the Critical Penetrating Power

It is seen from equation (16) that the average wavelength at which the J-discontinuity occurs is given, to a rough approximation, by $(a/8b)^{1/3}$ which is a constant for a given absorber. But this average wavelength is also a measure of the macroscopic complexion of the heterogeneous radiation. The occurrence of the J-discontinuity is thus conditional upon the availability and constancy of this complexion within the scope of the experiment and not upon the individual wavelengths constituting the incident beam. This is the reason why, with beams obtained in so many diverse ways, their critical penetrating power should turn out so constant, when measured with any substance.

J-step

Referring to our Fig. 2, if y'_Y be the abscissa of the discontinuity, then the corresponding ordinates belonging to the two straight lines represented by the equations (12) and (13) are given by

$$AX + \frac{B_X}{B_Y}(1-3\alpha)y'_Y$$

and

$$AX + \frac{B_X}{B_Y}(1-3\beta)y'_Y$$

respectively and their difference is

$$3\,\frac{B_X}{B_Y}(\beta-\alpha)y'_Y$$

This is the J-step.

The percentage step-up compared to y'_Y

$$= \left\{ 3\,\frac{B_X}{B_Y}(\beta-\alpha)y'_Y \times 100 \right\} / y'_Y = 300\,\frac{B_X}{B_Y}(\beta-\alpha).$$

Multiple discontinuities

The occurrence sometimes of more than one discontinuity in course of a single experiment, is in the light of the present analysis, suggestive of the existence of a corresponding number of D-points along the whole length of the curve $(y-x)$ represented by equation (3a) and brought under experimental observation. The a, b, c—values over different segments of the curve, are, in that case, different. We are, however, inclined to the view that the double discontinuities in Barkla's Fig. 3, p. 1041, is possibly due to D_X and D_Y being somewhat separated along x-direction.

CASE II

Relation between the mass-absorption coefficient and the fraction of the radiation absorbed, when the given incident beam is increasingly filtered

Let Z denote the fraction of the radiation cut off by the filter and $(\bar{\mu}/\rho)_X$ the mass-absorption coefficient of the transmitted beam, as measured with the testing substance X. Then the average wavelength λ of the transmitted beam is a function of Z. We can therefore write :

$$x = \psi(Z) \text{ or } Z = f(x), \text{ say} \tag{18}$$

where $x = \lambda^3$, as before.

Since the wavelength emerging from the filter is the wavelength incident on the testing substance we also have from (11a) near the D-point :

$$y_X - (\bar{\mu}/\rho)_X = A_X + B_X x(1 - 3K_X) \tag{19}$$

The required relation between $(\bar{\mu}/\rho)_X$ and Z will be obtained by eliminating x between equations (18) and (19). Although the exact form of the function f in (18) is not known, yet the conditions to be satisfied by it are quite definite and may be utilised in ascertaining the general run of the curve $(Z-x)$ in (18). The conditions are :

(i) As Z increases, x decreases.

(ii) As Z increases, dx/dZ decreases and hence dZ/dx increases numerically.

(iii) When $Z = 0$, $x = x_0$ (corresponding to the unfiltered radiation).

(iv) When $Z = 1$, x has a minimum value which is not zero.

These are illustrated in the above Fig. 3, where the curve O'PQ represents the equation $Z = f(x)$, and the curve RPS, represents equation (3a). The Z-axis and y-axis are represented by the L.H.S. and R.H.S. vertical lines respectively. The points Q and P, on the curve O' PQ correspond to 100 per cent and 50 per cent reduction in intensity respectively. As the actual experiments under consideration (Barkla's Figs. 10 and 11) were performed with a radiation filtered till the intensity was cut off by 50%, i.e., till Z increased from zero to 0.5, they are necessarily confined to the region of the diagram (Fig. 3), lying on the R.H.S. of P only. The D-point described in Case I previously and assumed present, is also shown (encircled) on the curve RPS and lying somewhere between P and S. The abscissa x is drawn on a scale somewhat exaggerated.

It may be pointed out that so far as the J-discontinuity is concerned, an exact knowledge of the function f is not essential. In view of the smallness of the

part O'P($\Delta\lambda = 0.1$ Å approx), the function may be regarded as practically linear. We may assume that the curve O'PQ is concurrent with a small portion of a parabola with a large latus rectum.

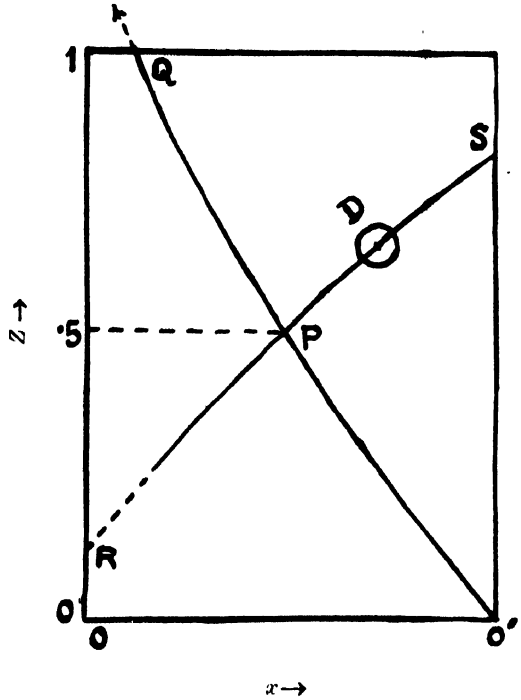

Fig. 3

Accordingly, for the curve O'PQ, we write

$$(a_1 - Z)^2 = b_1(x + \sigma) \tag{20}$$

where a_1, b_1 and σ are constants depending on the filtering material, b_1 being the latus rectum. This equation will be found to satisfy all the conditions, enumerated above. Thus

when $Z \to 1$(max), x is minimum in keeping with (iv), and assuming $Z << a_1$, $\frac{dZ}{dx}$ is negative in keeping with (i), since $\frac{dZ}{dx} = -\frac{b_1}{2(a_1 - Z)}$, also as Z increases, $\frac{dZ}{dx}$ increases in magnitude, in keeping with (ii) and further, when $Z = 0$, $x =$ max. $= x_0$, agreeing with (iii) so that from (20) we have,

$$a_1^2 = b_1(x_0 + \sigma) \qquad \ldots \quad (21)$$

Therefore, from (20) and (21), the equation to the curve O'PQ satisfying all the required conditions (i), (ii), (iii) and (iv), may be written down also as :

$$\left(\frac{a_1 - Z}{a_1}\right)^2 = \frac{x + \sigma}{x_0 + \sigma}$$

or $$\left(1-\frac{Z}{a_1}\right)^2 = \frac{x+\sigma}{x_0+\sigma}$$

$$\frac{2Z}{a_1} = \frac{x+\sigma}{x_0+\sigma} \quad \text{(approximat}$$

or $$Z = -\frac{a_1}{2}\cdot\frac{x-x_0}{x_0+\sigma} = f(x) \qquad \ldots (22)$$

The above is practically a linear relationship between Z and x. Hence the portion O'P of the curve is straight to the extent that the values of Z and a_1 conform to the assumed condition.

Taking O'P to be approximately straight and considering the short segments of the curve PS on either sides of D (supposed present) close to it, as linear, we may now eliminate x between the equations (19) and (22) and obtain :

$$y_X = \{A_X + B_X x_0(1-3K_X)\} - \left\{\frac{2}{a_1}(x_0+\sigma)B_X(1-3K_X)\right\} Z \qquad \ldots (23)$$

The required relation between y_X and Z is thus linear. But as the values of the constant K_X are different for the two sides of 'D' (*vide* Case *I*), the straight line in (23) splits up into two different (non-collinear) straight lines represented by

(i) $$y_X = \{A_X+B_X x_0(1-3K'_X)\} - \left\{\frac{2}{a_1}(x_0+\sigma)B_X(1-3K'_X)\right\} Z$$

and (ii) $$y_X = \{A_X+B_X x_0(1-3K''_X)\} - \left\{\frac{2}{a_1}(x_0+\sigma)B_X(1-3K''_X)\right\} Z \quad . \qquad (24)$$

where K'_X belongs to the left hand side of 'D' and K''_X to the right hand side in Fig. 3, or in other words K'_X corresponds to the shorter wavelength side of 'D' (and hence to larger values of Z) and K''_X to the longer wavelength side (and hence to smaller values of Z).

Now, $K_X(=e_X/\lambda)$ increases with λ for a filtered radiation, which is apparent from equation (7). Therefore, $K''_X > K'_X$. Hence the 'intercept' and the 'slope' of the straight line (i) are respectively greater than the corresponding quantities of the straight line (ii), the two stragiht lines being, of course, intervened by a discontinuity initiated by the D-point, exactly as in Case I. This is the J-discontinuity in Case II. The two straight lines agree in all essential features with those illustrated by Barkla in his Figs. 10 and 11, pp. 1047-1048.

It is evident that here also the J-discontinuity should be formed at the same critical penetrating power characteristic of the testing element, as in Case I and this corresponds to

$$x_j = \{-2b+\sqrt{4b^2+11ac}\}/22c$$

$$\approx \frac{a}{8b} \text{ (*vide* Eqns. 15 and 16)}$$

This feature, viz. the constancy of the critical penetrating power had been verified in Barkla's experiments.

The above treatment is a general one, in as much as it does not take into consideration the actual material of the filter; i.e., whether the material of the filter is the same as or different from that of the testing substance for measuring the mass-absorption coefficient. Analysis reveals that if there is a *D*-point, there should appear a discontinuity. In Barkla's experiments, however, discontinuity occurred only when the materials of the filter and the testing substance were identical, i.e., when both were aluminium or both copper. But the discontinuity was missing when the filter was aluminium and the testing substance copper. This according to the present view, only means that the *D*-point was either absent in that case, or was situated close to P (or S) but outside the segment PS (Fig. 3). The linear relation between $(\bar{\mu}/\rho)_{Cu}$ and Z_{Al} (Fig. 10, Barkla), suggests that the latter alternative should hold.

VERIFICATIONS

The fact that the theory advanced here has yielded results in general accord with the experimental findings of Barkla, amply bears out the correctness of the various assumptions made previously in that connection. A further justification may be had from an actual examination of the experimental data furnished below.

With the help of Barkla's Fig. 10 (lower diagram) and Fig. 11, together with our equations (21 and 24), it is possible to make the following computations :

(a) For Aluminium-Aluminimum pair,

$$K_1 \quad -K'_X \approx 0.057, \quad b_1/a_1 \approx 32, \quad a_1 > 1.58, \; b_1 \quad 50.5,$$

and since $$Z < 0.5, \quad Z^2/a_1^2 < 0.1.$$

(b) For Copper-Copper pair,

$$K''_X - K'_X \approx 0.093, \; b_1/a_1 \approx 17.7, \; a_1 > 2.17, \; b_1 > 38.5$$

and since $Z < 0.5$, $$\frac{Z^2}{} < 0.06$$

Absence of the J-discontinuity

We shall now discuss the circumstances leading to the non-appearance or absence of the *J*-discontinuity. Evidently, where the 'discrepancy' e is either zero or very small, the *D*-point is absent or practically so. Hence a discontinuity is not to be expected there. This is so

(i) with a beam of X-rays which is strictly monochromatic for which $e = 0$,

(ii) with a beam of heterogeneous X-rays which has been excessively filtered and consequently rendered more or less homogeneous, when e becomes insignificant and

(iii) with a method of measurement where the absorption coefficient of the filtered heterogeneous beam is determined arbitrarily from insufficient absorption—say, 25% instead of 50% as usual—within the testing substance, thus making e too small.

Case (ii) here, was nicely illustrated by Dunbar's (1925) failures and (iii) by Barkla's Fig. 8 (lower diagram). In all these cases, the graph $(y_X - y_Y)$ was found to be linear as expected from Equation (11).

There is a fourth case in which the dicontinuity may not appear. If the D-points do not fall within the range of wavelenths studied, but are yet not far away, only one side of the D-points can be considered. In such a case we should expect a single continuous straight line for the $(y_X - y_Y)$ graph, instead of two, intervened by the discontinuity. A fifth case for no discontinuity may arise if $\alpha = \beta$ (*vide* equation 12 and 13) when the two straight lines should merge into one, even with the D-point present within the experimental range. (See Barkla's Fig. 4. for unfiltered radiation and the upper diagram of Fig. 10.).

Barkla and his collaborators had observed that the J-discontinuity sometimes occurred and sometimes did not occur under what appeared to be essentially identical conditions. This may be expected when the beam of X-rays concerned is in a state of unsteadiness. Irregular changes in the structure of the radiation may, at times, render the plotted points misleading and unrealiable with a consequent obliteration of the J-step. (See Barkla's Fig. 2, p. 1040). Further it can be suggested that the 'discrepancy'-coefficients, a, b, c (in Equation 7) of an absorbing material may sometimes be affected by variations in some special factor or factors which have not been identified and controlled. The D-points may be sensitive to such variations, even when all other conditions are steady, appearing sometimes at the right places and sometimes moving beyond them. These considerations may explain, to some extent, the elusive nature of the J-discontinuity.

SUMMARY AND CONCLUSIONS

In the foregoing analysis of the J-phenomenon, we have accepted the hypothesis put forward by Barkla that a heterogeneous beam of X-rays behaves like an 'atmosphere' of radiation considered as an integrated whole for which its mass-absorption coefficient appears to be more fundamental than its individual wavelengths. It has been supposed that the mass-absorption coefficient of a heterogeneous beam of X-rays in an absorbing substance corresponds to some intermediate wavelength which the beam has somewhere inside the absorber. Since the softer constituents of the heterogeneous beam are increasingly filtered off as the beam passes through the absorber, this intermediate or the effective wavelength must needs be slightly less than the average wavelength. Expressing this 'discrepancy' between the effective wavelength and the average wavelength

as a suitable function of the latter, represented by (7), the *J*-discontinuity has been shown to be associated with the so-called *D*-point on the $(\bar{\mu}/\rho - \lambda^3)$-curve, where the rate of the variation of the slope is minimum (or zero as at a point of inflexion), the small segments of the curve lying on both sides of '*D*' and close to it, being regarded as straight lines. The position of the *J*-discontinuity, the critical mass-absorption coefficient for the discontinuity, the constancy of this critical value for a particular absorbing substance and its dependence on the material of the absorber and the magnitude of the *J*-step have been fully discussed in terms of the 'discrepancy' coefficients. Quantitative agreement has been shown in some cases between the results of the analysis and the experimental results of Barkla and his collaborators.

The absence of the *J*-discontinuity and the circumstances leading to it have been discussed in some details. Suggestions have also been made to explain the elusive nature of the *J*-discontinuity.

In conclusion, it is to be noted that Barkla's interpretation of the experimental results on the *J*-phenomenon in X-rays in terms of 'levels' of *X*-ray activity has not been considered in the present investigation. Barkla's fruitful idea that in the phenomena concerning a heterogeneous beam of X-rays, it is the average constitution of the beam that counts above every thing, is however recognised in the analysis detailed in this paper.

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